SECTION 4.4: INDETERMINATE FORMS AND L'HOPITAL'S RULE — WORKED EXAMPLES

① $\lim\limits_{x \to 0} \dfrac{e^{2x} - 1}{x} = \underline{\qquad}$? FORM: $0/0$

$f(x) = e^{2x} - 1$, $\quad f'(x) = 2e^{2x}$

$g(x) = x$, $\quad g'(x) = 1$

L'HOPITAL'S RULE: UNDER CERTAIN CONDITIONS:

$$\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f'(x)/g'(x).$$

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \lim_{x \to 0} \frac{2e^{2x}}{1} = \frac{2}{1} = 2$$

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② $\lim\limits_{x \to \infty} \dfrac{\ln x}{x} = \underline{\qquad}$? FORM: ∞/∞

use $f(x) = \ln x$, $\quad f'(x) = 1/x$.

$g(x) = x$, $\quad g'(x) = 1$.

$$\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{(1/x)}{1} = \frac{0}{1} = 0$$

(3) In this example, we must apply L'Hopital's Rule TWICE.

$\lim_{x \to -\infty} \left(\frac{x^2}{e^{-x}}\right) = \_\_\_\_\_ ?$ FORM: ∞/∞

Use $f(x) = x^2$, $f'(x) = 2x$
$g(x) = e^{-x}$, $g'(x) = -e^{-x}$

$\lim_{x \to -\infty} \left(\frac{x^2}{e^{-x}}\right) = \lim_{x \to -\infty} \left(\frac{2x}{-e^{-x}}\right) = \underline{\quad ?\quad}$

FORM Now is $-\infty/-\infty$

$f'(x) = 2x$, $f''(x) = 2$
$g'(x) = -e^{-x}$, $g''(x) = e^{-x}$

$\lim_{x \to -\infty} \left(\frac{2x}{-e^{-x}}\right) = \lim_{x \to -\infty} \left(\frac{2}{e^{-x}}\right) = 0$ since,

as $x \to -\infty$, $2 \to 2$ and $e^{-x} \to \infty$.

$\therefore \lim_{x \to -\infty} \left(\frac{x^2}{e^{-x}}\right) = 0$.

(4.) DEALING WITH FORM: $0 \cdot \infty$

TECHNIQUE 1: Write:

$$\lim_{x \to a} f(x)\,g(x) = \lim_{x \to a} \frac{f(x)}{\left(1/g(x)\right)} \quad \text{with Form } \frac{0}{0}.$$

TECHNIQUE 2: Write:

$$\lim_{x \to a} f(x)\,g(x) = \lim_{x \to a} \frac{g(x)}{\left(1/f(x)\right)} \quad \text{with Form } \frac{\infty}{\infty}$$

EXAMPLE (Solved using Technique 2):

$$\lim_{x \to \infty} e^{-x}\sqrt{x} = \underline{\qquad}? \quad \text{Form: } 0 \cdot \infty$$

Use $f(x) = e^{-x}$, $\frac{1}{f(x)} = \frac{1}{e^{-x}} = e^{x}$, $g(x) = \sqrt{x}$

$$\lim_{x \to \infty} e^{-x}\sqrt{x} = \lim_{x \to \infty} \left(\frac{\sqrt{x}}{e^{x}}\right) = \underline{\qquad}? \quad \text{Form: } \frac{\infty}{\infty}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} \;;\; \frac{d}{dx}(e^{x}) = e^{x}$$

$$\lim_{x \to \infty} \left(\frac{\sqrt{x}}{e^{x}}\right) = \lim_{x \to \infty} \frac{\left(\frac{1}{2\sqrt{x}}\right)}{e^{x}} = \lim_{x \to \infty} \frac{1}{2\sqrt{x}\,e^{x}} = 0$$

$$\therefore \lim_{x \to \infty} e^{-x}\sqrt{x} = 0$$

(5) EXAMPLE WITH FORM 0^0:

TECHNIQUE #1: Write $(f(x))^{g(x)} = e^{g(x)\ln f(x)}$.

If $g(x)\ln f(x) \to B$, Then

$$(f(x))^{g(x)} = e^{g(x)\ln f(x)} \longrightarrow e^{B} \text{ as a limit.}$$

TECHNIQUE #2: When considering $\lim\limits_{x\to a} (f(x))^{g(x)}$,

write $y = (f(x))^{g(x)}$.

Take $\ln(-)$ of both sides:

$\therefore \ln y = g(x)\ln(f(x))$ which has form $0\cdot(-\infty)$.

Then, if $\ln y \to B$, $y = e^{(\ln y)} \to e^{B}$.

EXAMPLE (solved using Technique 2):

$\lim\limits_{x\to 0^+} (\sin x)^x = \_\_\_\_$? FORM: 0^0.

write $y = (\sin x)^x$; $\ln y = x\ln(\sin x)$

which now has form $0\cdot(-\infty)$

$\therefore \ln y = \dfrac{\ln(\sin x)}{(1/x)}$ with form $\dfrac{-\infty}{\infty}$

$$\lim_{x\to 0^+}(\ln y) = \lim_{x\to 0^+}\frac{\ln(\sin x)}{(1/x)} = \lim_{x\to 0^+}\frac{\left(\frac{\cos x}{\sin x}\right)}{(-1/x^2)}$$

(5) cont.:

$$\lim_{x\to 0^+} \frac{(\cos x/\sin x)}{(-1/x^2)} = \lim_{x\to 0^+} \frac{\cot x}{(-1/x^2)} =$$

$$= \lim_{x\to 0^+} \frac{(1/\tan x)}{(-1/x^2)} = \lim_{x\to 0^+} \frac{-x^2}{\tan x} \text{ with form } \frac{0}{0}.$$

$$\frac{d}{dx}(\tan x) = \sec^2 x, \quad \frac{d}{dx}(-x^2) = -2x$$

$$\therefore \lim_{x\to 0^+} \frac{-x^2}{\tan x} = \lim_{x\to 0^+} \frac{-2x}{\sec^2 x} = \frac{(-2)\cdot(0)}{(1)^2} = 0.$$

$$\therefore \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \ln((\sin x)^x) = 0$$

$$\therefore \lim_{x\to 0^+} (\sin x)^x = \lim_{x\to 0^+} e^{(\ln(\sin x)^x)} =$$

$$= e^{\lim_{x\to 0^+}(\ln(\sin x)^x)} = e^0 = 1.$$

$$\therefore \lim_{x\to 0^+} (\sin x)^x = 1.$$

(6) EXAMPLE WITH FORM $\infty - \infty$

<u>TECHNIQUE</u>: MANIPULATE THE EXPRESSIONS (SAY by Adding Fractions into a single Fraction) to obtain Form $0/0$ or ∞/∞.

$$\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = \underline{\quad} ? \quad \text{Form: } \infty - \infty.$$

$$\frac{1}{\ln x} - \frac{1}{x-1} = \frac{(x-1) - \ln x}{(\ln x)(x-1)} \quad \text{with form: } 0/0$$

$$\frac{d}{dx}\left((\ln x)(x-1)\right) = \frac{1}{x}(x-1) + \ln x = 1 - 1/x + \ln x$$

$$\therefore \lim_{x \to 1^+} \left(\frac{(x-1) - \ln x}{(\ln x)(x-1)} \right) = \lim_{x \to 1^+} \left(\frac{1 - 1/x}{1 - 1/x + \ln x} \right) \leftarrow \text{BY L.R}$$

$$= \lim_{x \to 1^+} \left(\frac{\frac{(x-1)}{x}}{\frac{x-1+x\ln x}{x}} \right) = \lim_{x \to 1^+} \frac{x-1}{x-1+x\ln x}$$

$$= \lim_{x \to 1^+} \left(\frac{1}{1 - 0 + (\ln x + x \cdot \frac{1}{x})} \right) = \lim_{x \to 1^+} \left(\frac{1}{1+\ln x+1} \right)$$

BY L.R.

$$= \frac{1}{2}.$$

$$\therefore \lim_{x \to 1^+} \left(1/\ln x - \frac{1}{x-1} \right) = 1/2.$$

BEWARE:

DO NOT BLINDLY APPLY L'HOPITAL'S RULE WITHOUT CHECKING FOR AN APPROPRIATE INDETERMINATE FORM:

$$\lim_{x \to \pi^-} \left(\frac{\sin x}{(1-\cos x)} \right) = \underline{\qquad} ?$$

INCORRECT WORK:

$$\frac{d}{dx}(\sin x) = \cos x \, , \; \frac{d}{dx}(1-\cos x) = \sin x$$

$$\lim_{x \to \pi^-} \cos x = -1 \, , \; \lim_{x \to \pi^-} (\sin x) = 0$$

$$\lim_{x \to \pi^-} \left(\frac{\sin x}{(1-\cos x)} \right) = \lim_{x \to \pi^-} \frac{\cos x}{\sin x} = -\infty$$

WRONG!

CORRECT WORK:

$\lim_{x \to \pi^-} \left(\frac{\sin x}{(1-\cos x)} \right)$ is not of form $0/0$ since

$$\lim_{x \to \pi^-} (1-\cos x) = 1-(-1) = 2$$

$$\therefore \lim_{x \to \pi^-} \left(\frac{\sin x}{(1-\cos x)} \right) = \frac{\lim_{x \to \pi^-} \sin x}{\lim_{x \to \pi^-} (1-\cos x)} = \frac{0}{2} = 0$$